Last time: Sets and Set Operations

Venn Diagrams ← (Just for Intuition... NOT PROOFS)

Recall: For set S, T we defined:

$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$

$$S \cup T = \{x : x \in S \text{ or } x \in T\}$$

$$S \setminus T = \{x : x \in S \text{ but } x \notin T\}$$

$$S \times T = \{(x,y) : x \in S \text{ and } y \in T\}$$

$S \subseteq T$ when $\forall x \in S\,[x \in T] \equiv \forall x\,[x \in S \to x \in T]$

$S = T$ when $\forall x\,[x \in S \leftrightarrow x \in T]$ ←

Prop: Let S and T be sets. Then $S \cup T = T \cup S$.

Pf: Let S and T be sets. ← state assumptions. Remarks = red.

(Remark: we will show $S \cup T \subseteq T \cup S$ and $T \cup S \subseteq S \cup T$) ←

signals first step

To see $S \cup T \subseteq T \cup S$, we let $x \in S \cup T$ be arbitrary. ← assumption for conditional proof

Thus $x \in S$ or $x \in T$ by the definition of set union. ← justification.

As a result, $x \in T$ or $x \in S$ by commutativity of "or". ← conclusion.

Hence $x \in T \cup S$ by definition of set union. We now see $x \in S \cup T$ implies $x \in T \cup S$, so $S \cup T \subseteq T \cup S$ by definition of the subset relation.

To see $T \cup S \subseteq S \cup T$, we let $x \in T \cup S$ be arbitrary. Thus $x \in T$ or $x \in S$ by definition of set union. We thus obtain $x \in S$ or $x \in T$ by commutativity of

"or". Hence $x \in S \cup T$ by definition of set union.
We now see $x \in T \cup S$ implies $x \in S \cup T$, so $T \cup S \subseteq S \cup T$.

Now we have $S \cup T \subseteq T \cup S$ and $T \cup S \subseteq S \cup T$, so

$S \cup T = T \cup S$ as originally claimed. ▯

Same pattern.

Last line: Confirms (to the reader) we think we're done proving!

Point: Every good proof follows these rules:

① Clearly written (mathematical) English.

② Every claim we make comes with a justification via "previous results," definitions, and basic logic.

Rule of Thumb: Treat your reader like a compiler, and the language you use like a programming language. Must follow the established syntax and semantics!

NB: To prove two sets are equal, we prove they are mutually subsets of one another...

I.e. to prove $A = B$, we show $A \subseteq B$ and $B \subseteq A$.

Prop: Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

pf: Let A, B, and C be sets. Suppose $A \subseteq B$ and $B \subseteq C$. (WTS: $A \subseteq C$ i.e. $\forall x [x \in A \to x \in C]$)

↑ "want to show"

Let $x \in A$ be arbitrary. (WTS: $x \in C$).

As $A \subseteq B$ and $x \in A$, we have $x \in B$ by definition of the subset relation.

As $B \subseteq C$ and $x \in B$, we have $x \in C$ by definition of the subset relation.

($\forall y [y \in A \to y \in B]$, $x \in A$ $\therefore x \in B$ "Quantified Modus ponens")

As $x \in A$ was arbitrary, we have $A \subseteq C$ by definition of the subset relation. ☒

Prop: For all sets $A, B,$ and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Pf: Let $A, B,$ and C be sets.

($\subseteq$): Let $x \in A \cap (B \cup C)$ be arbitrary. By definition of intersection we have $x \in A$ and $x \in B \cup C$. By definition of set union we have both $x \in A$ and either $x \in B$ or $x \in C$.

(translates: $[x \in A] \wedge ([x \in B] \vee [x \in C])$)

Thus, by distribution of "and" over "or", we have either both $x \in A$ and $x \in B$, or both $x \in A$ and $x \in C$. Thus we have either $x \in A \cap B$ or $x \in A \cap C$ by definition of set intersection. Hence $x \in (A \cap B) \cup (A \cap C)$ by definition of set union. As x was arbitrary, we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

($\supseteq$): Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. Thus either $x \in A \cap B$ or $x \in A \cap C$ by definition of set union. By definition of set intersection, either

both $x \in A$ and $x \in B$ or both $x \in A$ and $x \in C$.

Thus both $x \in A$ and either $x \in B$ or $x \in C$ by distribution of "and" over "or". Thus both $x \in A$ and $x \in B \cup C$ by definition of set union. Finally, $x \in A \cap (B \cup C)$ by definition of set intersection.

As x was arbitrary, we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence as both $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, we have shown $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ as desired. ☑

$$\forall_{x \in A}[x \in B] \equiv \forall_x[(x \in A) \to (x \in B)]$$

Recall: $\mathbb{P}(S) = \{T : T \subseteq S\}$ ←

Prop: Let A and B be sets.
If $A \subseteq B$, then $\mathbb{P}(A) \subseteq \mathbb{P}(B)$.

pf: Let A and B be sets and suppose $A \subseteq B$. Let $X \in \mathbb{P}(A)$ be arbitrary. By definition of the powerset, we have $X \subseteq A$; on the other hand, $A \subseteq B$, so $X \subseteq B$ by a* previous proposition. Hence $X \in \mathbb{P}(B)$ by definition of powerset. As $X \in \mathbb{P}(A)$ was arbitrary, we have $\mathbb{P}(A) \subseteq \mathbb{P}(B)$ as claimed ☑

Propositions to try proving for homework:

Let A, B, C, D be sets.

① $A \cap B = B \cap A$.

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

③ $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

↳ Q: Does $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$?

④ Is $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$?

Is $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$?